SADLER MATHEMATICS METHODS UNIT 2

WORKED SOLUTIONS

Chapter 5 Rates of change

Exercise 5A

Question 1

a From A to B, from D to F

b From B to D, from F to I

c At B,D, F and H

Question 2

Graph	True Statements					
I	B	C	D	E	F	G
II	A					
III	H					
IV	A	D	G			
V	A	D	H			
VI	C	F				
VII	A	E	G			
VIII	A	B	F	G		
IX	A	E	F	G		
X	B	F	H			

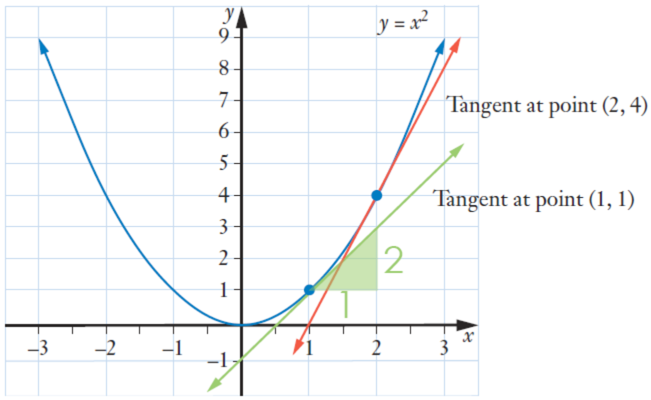
Question 3

a C, E, H, K, M, O

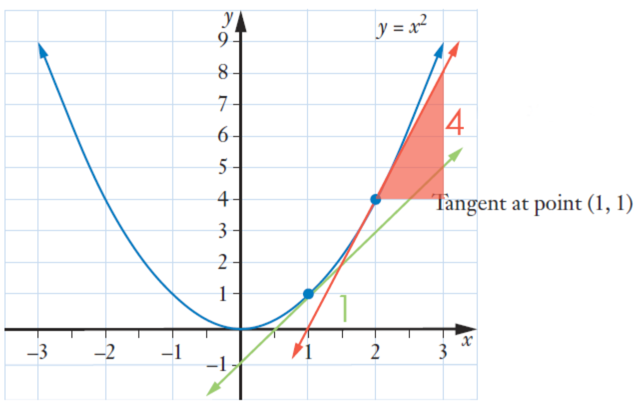
b A, B, I, J, N, P

c D, F, G, L

Question 4



a By considering the height of the green triangle, we can see the tangent has a gradient of 2



b By considering the height of the orange triangle, we can see the tangent has a gradient of 4

c The tangent at (0, 0) is the *x*-axis so the gradient is 0.

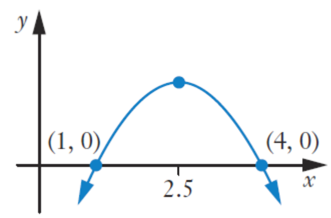
d By symmetry, the gradient at $x = -1$ is –2.

e By symmetry, the gradient at $x = -2$ is –4.

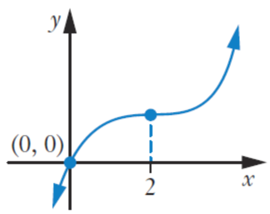
f The curve $y = x^2 + 3$ is exactly the same shape as $y = x^2$ so the gradient is 2.

g) The curve $y = (x-2)^2$ is exactly the same shape as $y = x^2$ but translated 2 units right.
The gradient at $x = 3$ on $y = (x-2)^2$ is the same as the gradient as $x = -1$ on $y = x^2$, i.e. 2.

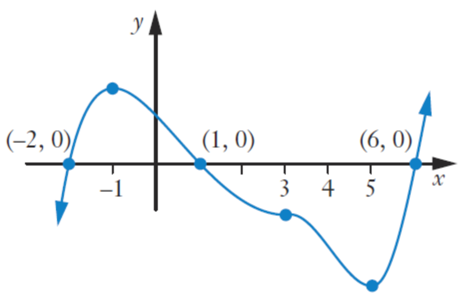
Question 5



Question 6



Question 7



Question 1

Point P	Point Q	Gradient of chord PQ
(2, 4)	(4, 16)	$\frac{16-4}{4-2}=6$
(2, 4)	(3, 9)	$\frac{9-4}{3-2}=5$
(2, 4)	(2.5, 6.25)	$\frac{6.25-4}{2.5-2}=4.5$
(2, 4)	(2.1, 4.41)	$\frac{4.41-4}{2.1-2}=4.1$
(2, 4)	(2.01, 4.0401)	$\frac{4.0401-4}{2.01-2}=4.01$
(2, 4)	(2.001, 4.004001)	$\frac{4.004001-4}{2.001-2}=4.001$
(2, 4)	(2.0001, 4.00040001)	$\frac{4.00040001-4}{2.0001-2}=4.0001$

The gradient at $y=x^2$ at $x=2$ is 4.

Question 2

Point P	Point Q	Gradient of chord PQ
(3, 9)	(4, 16)	$\frac{16-9}{4-3}=7$
(3, 9)	(3.5, 12.25)	$\frac{12.25-9}{3.5-3}=6.5$
(3, 9)	(3.1, 9.61)	$\frac{9.61-9}{3.1-3}=6.1$
(3, 9)	(3.01, 9.0601)	$\frac{9.0601-9}{3.01-3}=6.01$
(3, 9)	(3.001, 9.006001)	$\frac{9.006001-9}{3.001-3}=6.001$

The gradient at $y=x^2$ at $x=3$ is 6.

Point P	Point Q	Gradient of chord PQ
(4, 16)	(5, 25)	$\frac{25-16}{5-4}=9$
(4, 16)	(4.5, 20.25)	$\frac{20.25-16}{4.5-4}=8.5$
(4, 16)	(4.1, 16.81)	$\frac{16.81-16}{4.1-4}=8.1$
(4, 16)	(4.01, 16.0801)	$\frac{16.0801-16}{4.01-4}=8.01$
(4, 16)	(4.001, 16.008001)	$\frac{16.008001-16}{4.001-4}=8.001$

The gradient at $y=x^2$ at $x=4$ is 8.

Point P	Point Q	Gradient of chord PQ
(5, 25)	(6, 36)	$\frac{36-25}{6-5}=11$
(5, 25)	(5.5, 30.25)	$\frac{30.25-25}{5.5-5}=10.5$
(5, 25)	(5.1, 26.01)	$\frac{26.01-25}{5.1-5}=10.1$
(5, 25)	(5.01, 25.1001)	$\frac{25.1001-25}{5.01-5}=10.01$
(5, 25)	(5.001, 25.010001)	$\frac{25.010001-25}{5.001-5}=10.001$

The gradient at $y=x^2$ at $x=5$ is 10.

For $y=x^2$

x	0	1	2	3	4	5
gradient	0	2	4	6	8	10

The gradient at $x=a$ on the curve $y=x^2$ is $2a$.

Question 3

Point P	Point Q	Gradient of chord PQ
(2, 12)	(3, 27)	$\frac{27-12}{3-2}=15$
(2, 12)	(2.5, 18.75)	$\frac{18.75-12}{2.5-2}=13.5$
(2, 12)	(2.1, 13.23)	$\frac{13.23-12}{2.1-2}=12.3$
(2, 12)	(2.01, 12.1203)	$\frac{12.1203-12}{2.01-2}=12.03$
(2, 12)	(2.001, 12.012003)	$\frac{12.012003-12}{2.001-2}=12.003$

The gradient at $y=3x^2$ at $x=2$ is 12.

Point P	Point Q	Gradient of chord PQ
(3, 27)	(4, 48)	$\frac{48-27}{4-3}=21$
(3, 27)	(3.5, 36.75)	$\frac{36.75-27}{3.5-3}=19.5$
(3, 27)	(3.1, 28.83)	$\frac{28.83-27}{3.1-3}=18.3$
(3, 27)	(3.01, 27.1803)	$\frac{27.1803-27}{3.01-3}=18.03$
(3, 27)	(3.001, 27.018003)	$\frac{27.018003-27}{3.001-3}=18.003$

The gradient at $y=3x^2$ at $x=3$ is 18.

Point P	Point Q	Gradient of chord PQ
(4, 48)	(5, 125)	$\frac{125-48}{5-4}=77$
(4, 48)	(4.5, 60.75)	$\frac{60.75-48}{4.5-4}=25.5$
(4, 48)	(4.1, 50.43)	$\frac{50.43-48}{4.1-4}=24.3$
(4, 48)	(4.01, 48.2403)	$\frac{48.2403-48}{4.01-4}=24.03$
(4, 48)	(4.001, 48.024003)	$\frac{48.024003-48}{4.001-4}=24.003$

The gradient at $y=3x^2$ at $x=4$ is 24.

For $y=3x^2$

x	0	1	2	3	4	5
gradient	0	6	12	18	24	30

The gradient at $x=a$ on the curve $y=3x^2$ is $6a$.

Question 1

$$\begin{aligned}\text{Gradient at } P(x,4x^2) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{4(x+h)^2-4x^2}{h}\\ &= \lim_{h\to 0}\frac{4x^2+8xh+4h^2-4x^2}{h}\\ &= \lim_{h\to 0}\frac{8xh+4h^2}{h}\\ &= \lim_{h\to 0} 8x+4h\\ &= 8x\end{aligned}$$

Question 2

$$\begin{aligned}\text{Gradient at } P(x,2x^3) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{2(x+h)^3-2x^3}{h}\\ &= \lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+2h^3-2x^3}{h}\\ &= \lim_{h\to 0}\frac{6x^2h+6xh^2+2h^3}{h}\\ &= \lim_{h\to 0} 6x^2+6xh+2h^2\\ &= 6x^2\end{aligned}$$

Question 3

$$
\begin{aligned}
\text{Gradient at } P(x, x^4) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^4-x^4}{h} \\
&= \lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} \\
&= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} \\
&= \lim_{h\to 0} 4x^3+6x^2h+4xh^2+h^3 \\
&= 4x^3
\end{aligned}
$$

Exercise 5D

Question 1

$$\frac{dy}{dx} = 2x$$

Question 2

$$\frac{dy}{dx} = 3x^2$$

Question 3

$$\frac{dy}{dx} = 1$$

Question 4

$$\frac{dy}{dx} = 4x^3$$

Question 5

$$\frac{dy}{dx} = 0$$

Question 6

$$\frac{dy}{dx} = 12x$$

Question 7

$$\frac{dy}{dx}=24x^3$$

Question 8

$$\frac{dy}{dx}=7$$

Question 9

$$\frac{dy}{dx}=16$$

Question 10

$$\frac{dy}{dx}=14x^6$$

Question 11

$$\frac{dy}{dx}=14x$$

Question 12

$$\frac{dy}{dx}=9$$

Question 13

$$\frac{dy}{dx}=\frac{2x}{10}=\frac{x}{5}$$

Question 14

$$\frac{dy}{dx} = \frac{12x^5}{3} = 4x^5$$

Question 15

$$\frac{dy}{dx} = \frac{18x^5}{2} = 9x^5$$

Question 16

$$\frac{dy}{dx} = \frac{14x^6}{7} = 2x^6$$

Question 17

$$\frac{dy}{dx} = 8x$$

Question 18

$$\frac{dy}{dx} = 20x^3$$

Question 19

$$\frac{dy}{dx} = 24x^2$$

Question 20

$$\frac{dy}{dx} = 0$$

Question 21

$$\frac{dy}{dx} = 7x^6$$

Question 22

$$\frac{dy}{dx} = 24x^5$$

Question 23

$$\frac{dy}{dx} = 18x$$

Question 24

$$\frac{dy}{dx} = 5$$

Question 25

$$f'(x) = 0$$

Question 26

$$f'(x) = 18x^2$$

Question 27

$$f'(x) = 32x^3$$

Question 28

$f'(x) = 15x^4$

Question 29

$f'(x) = 6x^5$

Question 30

$f'(x) = 42x^6$

Question 31

$f'(x) = 16x^3$

Question 32

$f'(x) = 10$

Question 33

$$\frac{dy}{dx} = 4x$$

at $x = 3$,

$$\begin{aligned} \frac{dy}{dx} &= 4(3) \\ &= 12 \end{aligned}$$

Question 34

$$\frac{dy}{dx} = 12x^2$$

at $x = 1$,

$$\frac{dy}{dx} = 12(1)^2$$
$$= 12$$

Question 35

$$\frac{dy}{dx} = 12x^2$$

at $x = -1$,

$$\frac{dy}{dx} = 12(-1)^2$$
$$= 12$$

Question 36

$$\frac{dy}{dx} = 5x^4$$

at $x = 2$,

$$\frac{dy}{dx} = 5(2)^4$$
$$= 80$$

Question 37

$\frac{dy}{dx} = 7$ at all points

Question 38

$$\frac{dy}{dx} = 10x$$

at $x = -2,$

$$\frac{dy}{dx} = 10(-2)$$
$$= -20$$

Question 39

$$\frac{dy}{dx} = 0.5x$$

at $x = 4,$

$$\frac{dy}{dx} = 0.5(4)$$
$$= 2$$

Question 40

$$\frac{dy}{dx} = \frac{2x}{5}$$

at $x = 2,$

$$\frac{dy}{dx} = \frac{2(2)}{5}$$
$$= 0.8$$

Question 41

$$\frac{dy}{dx} = 4x^3 = 4$$
$$x^3 = 1$$
$$x = 1$$

at $x = 1, y = (1)^4 = 1$

$\therefore (1,1)$

Question 42

$$\frac{dy}{dx} = 3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1$$
at $x = -1, y = (-1)^3 = -1$

at $x = 1, y = (1)^3 = 1$

$\therefore (-1, -1)$ and $(1, 1)$

Question 43

$$\frac{dy}{dx} = 6x = 9$$
$$x = 1.5$$
at $x = 1.5, y = 3(1.5)^2 = 6.75$

$\therefore (1.5,\ 6.75)$

Question 44

$$\frac{dy}{dx} = 6x^2 = 1.5$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\frac{1}{2}$$
at $x = -\frac{1}{2}, y = 2(-\frac{1}{2})^3 = -\frac{1}{4}$

at $x = \frac{1}{2}, y = 2(\frac{1}{2})^3 = \frac{1}{4}$

$\therefore (-\frac{1}{2}, -\frac{1}{4})$ and $(\frac{1}{2}, \frac{1}{4})$

Question 45

$$\frac{dy}{dx} = 6x^5 = 6$$

$$x^5 = 1$$

$$x = 1$$

at $x = 1, y = (1)^6 = 1$

$\therefore (1, 1)$

Question 46

$$\frac{dy}{dx} = 6x^5 = -6$$

$$x^5 = -1$$

$$x = -1$$

at $x = -1, y = (-1)^6 = 1$

$\therefore (-1, 1)$

Question 47

$$\frac{dy}{dx} = 6x^2$$

at $x = 1, y\dfrac{dy}{dx} = 6(1)^2 = 6$

$y = 6x + c$

$2 = 6(1) + c$

$c = -4$

$\therefore$ equation of tangent $y = 6x - 4$

Question 48

$\frac{dy}{dx} = 6x$

at $x = -1, y\frac{dy}{dx} = 6(-1) = -6$

$y = -6x + c$

$3 = -6(-1) + c$

$c = -3$

$\therefore$ equation of tangent $y = -6x - 3$

Question 49

$\frac{dy}{dx} = 10x$

at $x = 2, y\frac{dy}{dx} = 10(2) = 20$

$y = 20x + c$

$20 = 20(2) + c$

$c = -20$

$\therefore$ equation of tangent $y = 20x - 20$

Question 50

$\frac{dy}{dx} = 10x$

at $x = -2, y\frac{dy}{dx} = 10(-2) = -20$

$y = -20x + c$

$20 = -20(-2) + c$

$c = -20$

$\therefore$ equation of tangent $y = -20x - 20$

Question 51

$$\frac{dy}{dx} = 2x^3$$

at $x = 2, y\frac{dy}{dx} = 2(2)^3 = 16$

$y = 16x + c$

$8 = 16(2) + c$

$c = -24$

$\therefore$ equation of tangent $y = 16x - 24$

Question 52

$$\frac{dy}{dx} = \frac{x^2}{2}$$

at $x = 6, y\frac{dy}{dx} = \frac{(6)^2}{2} = 18$

$y = 18x + c$

$36 = 18(6) + c$

$c = -72$

$\therefore$ equation of tangent $y = 18x - 72$

Question 53

a $f(2) = 3(2)^3 = 24$

b $f(-1) = 3(-1)^3 = -3$

c $f'(x) = 9x^2$

d $f'(2) = 9(2)^2 = 36$

Question 54

a $f(2) = 1.5(2)^2 = 6$

b $f(4) = 1.5(4)^2 = 24$

c $f'(x) = 3x$

d $f'(2) = 3(2) = 6$

Question 55

a at $x = 2,\ y = 2(2)^3 = 16$

at $x = 5,\ y = 2(5)^3 = 250$

$\therefore\ y$ changes by 234

b $\dfrac{234}{3} = 78$ units per unit change in x

c $\dfrac{dy}{dx} = 6x^2$

at $x = 2$

$\dfrac{dy}{dx} = 6(2)^2 = 24$

d $\dfrac{dy}{dx} = 6x^2$

at $x = 5$

$\dfrac{dy}{dx} = 6(5)^2 = 150$

Question 56

$8x^2 = 8x + 16$

By classpad or

$8x^2 - 8x - 16 = 0$

$8(x^2 - x - 2) = 0$

$8(x-2)(x+1) = 0$

$\therefore x = -1, x = 2$

at $x = -1, y = 8(-1) + 16 = 8 \Rightarrow (-1, 8)$

at $x = 2, y = 8(2) + 16 = 32 \Rightarrow (2, 32)$

Gradient of $y = 8x^2$ at points of intersection

$\frac{dy}{dx} = 16x$

at $x = -1, \frac{dy}{dx} = 16(-1) = -16$

at $x = 2, \frac{dy}{dx} = 16(2) = 32$

Question 57

$x^3 = 4x$

By classpad or

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x(x-2)(x+2) = 0$

$\therefore x = -2, x = 0, x = 2$

at $x = -2, y = 4(-2) = -8 \Rightarrow (-2, -8)$

at $x = 0, y = 4(0) = 0 \Rightarrow (0, 0)$

at $x = 2, y = 4(2) = 8 \Rightarrow (2, 8)$

Gradient of $y = x^3$ at points of intersection

$\frac{dy}{dx} = 3x^2$

at $x = -2, \frac{dy}{dx} = 3(-2)^2 = 12$

at $x = 0, \frac{dy}{dx} = 3(0)^2 = 0$

at $x = 2, \frac{dy}{dx} = 3(2)^2 = 12$

Question 58

$\frac{dy}{dx} = 4ax^3$

at $x = 3$

$$\begin{aligned}\frac{dy}{dx} &= 4a(3)^3 = 2 \\ 108a &= 2 \\ a &= \frac{1}{54}\end{aligned}$$

$y = \frac{1}{54}x^4$

$$\begin{aligned}b &= \frac{1}{54}(3)^4 \\ &= 1.5\end{aligned}$$

Question 59

$y = 2x + 3 \Rightarrow m = 2$

perpendicular gradient $= -\dfrac{1}{2}$

$\dfrac{dy}{dx} = 3ax^2$

at $x = -1$,

$$\begin{aligned}\frac{dy}{dx} &= 3a(-1)^2 = -\frac{1}{2}\\ 3a &= -\frac{1}{2}\\ a &= -\frac{1}{6}\end{aligned}$$

$$\begin{aligned}y &= -\frac{1}{6}x^3\\ b &= -\frac{1}{6}(-1)^3\\ &= \frac{1}{6}\end{aligned}$$

Exercise 5E

Question 1

$$\frac{dy}{dx} = 2x + 3$$

Question 2

$$\frac{dy}{dx} = 3x^2 - 4$$

Question 3

$$\frac{dy}{dx} = 12x - 21x^2$$

Question 4

$$\frac{dy}{dx} = 12x^3 + 6x^2 - 5$$

Question 5

$$\frac{dy}{dx} = 7 + 2x$$

Question 6

$$\frac{dy}{dx} = 12x - 3$$

Question 7

$$\frac{dy}{dx} = 8x + 7$$

Question 8

$$\frac{dy}{dx} = 15x^2 - 8x$$

Question 9

$$\frac{dy}{dx} = 20x^3 - 3$$

Question 10

$$\frac{dy}{dx} = 4x + 7$$

Question 11

$$\frac{dy}{dx} = -6x + 7$$

Question 12

$$\frac{dy}{dx} = 1 + 2x + 3x^2 + 4x^3$$

Question 13

$$\frac{dy}{dx} = -4 + 6x - 6x^2 + 4x^3$$

Question 14

$$\frac{dy}{dx} = 3x^2 - 6x$$

at $x = 1$

$$\frac{dy}{dx} = 3(1)^2 - 6(1) = -3$$

Question 15

$$\frac{dy}{dx} = 6x^2$$

at $x = -2$

$$\frac{dy}{dx} = 6(-2)^2 = 24$$

Question 16

$$\frac{dy}{dx} = 3x^2 - 2x$$

at $x = 3$

$$\frac{dy}{dx} = 3(3)^2 - 2(3) = 21$$

Question 17

$$\frac{dy}{dx} = 3 - 6x^2 + 4x^3$$

at $x = 2$

$$\frac{dy}{dx} = 3 - 6(2)^2 + 4(2)^3 = 11$$

Question 18

$$\frac{dy}{dx} = 2x + 3$$

at $x = 2$

$$\frac{dy}{dx} = 2(2) + 3 = 7$$

$$y = 7x + c$$

$$10 = 7(2) + c$$

$$c = -4$$

$\therefore$ equation of tangent $y = 7x - 4$

Question 19

$$\frac{dy}{dx} = 4x - 7$$

at $x = 5$

$$\frac{dy}{dx} = 4(5) - 7 = 13$$

$$y = 13x + c$$

$$15 = 13(5) + c$$

$$c = -50$$

$\therefore$ equation of tangent $y = 13x - 50$

Question 20

$$\frac{dy}{dx} = 3x^2 - 10x$$

at $x = 4$

$$\frac{dy}{dx} = 3(4)^2 - 10(4) = 8$$

$$y = 8x + c$$

$$-2 = 8(4) + c$$

$$c = -34$$

$\therefore$ equation of tangent $y = 8x - 34$

Question 21

$$\frac{dy}{dx} = 20x^3 - 20x^4$$

at $x = 1$

$$\frac{dy}{dx} = 20(1)^3 - 20(1)^4 = 0$$

A gradient of 0 indicates a horizontal line

$\therefore$ equation of tangent $y = 1$

Question 22

$$\frac{dy}{dx} = 3x^2 + 12x - 10 = 5$$

$$3x^2 + 12x - 15 = 0$$

$$3(x^2 + 4x - 5) = 0$$

$$3(x+5)(x-1) = 0$$

$$x = -5, x = 1$$

at $x = -5$

$$y = (-5)^3 + 6(-5)^2 - 10(-5) + 1 = 76$$

at $x = 1$

$$y = (1)^3 + 6(1)^2 - 10(1) + 1 = -2$$

$\therefore$ coordinates $(-5,\ 76)$ and $(1,\ -2)$

Question 23

$x^2 - 2x - 15 = 0$

$(x+3)(x-5) = 0$

$x = -3, x = 5$

$\therefore$ coordinates (–3, 0) and (5, 0)

$\frac{dy}{dx} = 2x - 2$

at $x = -3$

$\frac{dy}{dx} = 2(-3) - 2 = -8$

at $x = 5$

$\frac{dy}{dx} = 2(5) - 2 = 8$

Question 24

$3y = 9x - 1$

$y = 3x - \frac{1}{3} \Rightarrow m = 3$

$\frac{d}{dx}(x^2 - 7x) = 2x - 7$

$2x - 7 = 3$

$2x = 10$

$x = 5$

at $x = 5, y = 5^2 - 7(5) = -10$

$\therefore$ coordinates (5, –10)

Question 25

$y = 2x + 3 \Rightarrow m = 2$

$$\frac{d}{dx}(x^3 + 3x^2 - 7x - 1) = 3x^2 + 6x - 7$$

$$3x^2 + 6x - 7 = 2$$

$$3x^2 + 6x - 9 = 0$$

$$3(x^2 + 2x - 3) = 0$$

$$3(x + 3)(x - 1) = 0$$

$$x = -3, x = 1$$

at $x = -3$

$y = (-3)^3 + 3(-3)^2 - 7(-3) - 1 = 20$

at $x = 1$

$y = (1)^3 + 3(1)^2 - 7(1) - 1 = -4$

$\therefore$ coordinates $(-3,\ 20)$ and $(1,\ -4)$

Question 1

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

Question 2

$$y = \frac{1}{x} = x^{-1}$$

$$\frac{dy}{dx} = -1x^{-2}$$

$$= -\frac{1}{x^2}$$

Question 3

$$y = \frac{3}{x} = 3x^{-1}$$

$$\frac{dy}{dx} = -1.3x^{-2}$$

$$= -\frac{3}{x^2}$$

Question 4

$$\frac{dy}{dx} = \frac{1}{2}6x^{-\frac{1}{2}}$$

$$= \frac{3}{\sqrt{x}}$$

Question 5

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3}6x^{-\frac{2}{3}} \\ &= \frac{2}{x^{\frac{2}{3}}} \\ &= \frac{2}{\sqrt[3]{x^2}}\end{aligned}$$

Question 6

$$\begin{aligned}y &= \sqrt{x^3} \\ &= (x^3)^{\frac{1}{2}} \\ &= x^{\frac{3}{2}} \\ \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{3\sqrt{x}}{2}\end{aligned}$$

Question 7

$$\begin{aligned}y &= 2x^{\frac{1}{3}} \\ \frac{dy}{dx} &= \frac{1}{3}2x^{-\frac{2}{3}} \\ &= \frac{2}{3x^{\frac{2}{3}}} \\ &= \frac{2}{3\sqrt[3]{x^2}}\end{aligned}$$

Question 8

$$
\begin{aligned}
y &= x^{-3} \\
\frac{dy}{dx} &= -3x^{-4} \\
&= -\frac{3}{x^4}
\end{aligned}
$$

Question 9

$$
\begin{aligned}
y &= x^{-4} \\
\frac{dy}{dx} &= -4x^{-5} \\
&= -\frac{4}{x^5}
\end{aligned}
$$

Question 10

$$
\begin{aligned}
y &= 2x^{-3} \\
\frac{dy}{dx} &= -3 \times 2x^{-4} \\
&= -\frac{6}{x^4}
\end{aligned}
$$

Question 11

$$
\begin{aligned}
y &= 5x^{-4} \\
\frac{dy}{dx} &= -4 \times 5x^{-5} \\
&= -\frac{20}{x^5}
\end{aligned}
$$

Question 12

$$\begin{aligned} y &= x^2 + x^{\frac{1}{2}} \\ \frac{dy}{dx} &= 2x + \frac{1}{2}x^{-\frac{1}{2}} \\ &= 2x + \frac{1}{2\sqrt{x}} \end{aligned}$$

Question 13

$$\begin{aligned} y &= 3x^2 - 4x^{\frac{1}{2}} \\ \frac{dy}{dx} &= 6x - \frac{1}{2}4x^{-\frac{1}{2}} \\ &= 6x - \frac{2}{\sqrt{x}} \end{aligned}$$

Question 14

$$\begin{aligned} y &= x + x^{-1} \\ \frac{dy}{dx} &= 1 + (-1)x^{-2} \\ &= 1 - \frac{1}{x^2} \end{aligned}$$

Question 15

$$\begin{aligned} y &= x^2 - x^{-2} \\ \frac{dy}{dx} &= 2x - (-2)x^{-3} \\ &= 2x + \frac{2}{x^3} \end{aligned}$$

Question 16

$$y = x^{\frac{1}{2}} + 3x^{-1}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + (-1)3x^{-2}$$

$$= \frac{1}{2\sqrt{x}} - \frac{3}{x^2}$$

Question 17

$$y = x^2 + x + 1 + x^{-1} + x^{-2}$$

$$\frac{dy}{dx} = 2x + 1 + (-1)x^{-2} + (-2)x^{-3}$$

$$= 2x + 1 - \frac{1}{x^2} - \frac{2}{x^3}$$

Question 18

$$f(x) = 2x^{-1}$$

$$f'(x) = (-1)2x^{-2}$$

$$= -\frac{2}{x^2}$$

Question 19

$$f(x) = 3x^{-\frac{1}{2}}$$

$$f'(x) = (-\frac{1}{2})3x^{-\frac{3}{2}}$$

$$= -\frac{3}{2x^{\frac{3}{2}}}$$

$$= -\frac{3}{2\sqrt{x^3}}$$

Question 20

$$f(x) = 6x^{-\frac{1}{3}}$$
$$f'(x) = (-\frac{1}{3})6x^{-\frac{4}{3}}$$
$$= -\frac{2}{x^{\frac{4}{3}}}$$
$$= -\frac{2}{\sqrt[3]{x^4}}$$

Question 21

$$f(x) = x^{-\frac{1}{3}}$$
$$f'(x) = (-\frac{1}{3})x^{-\frac{4}{3}}$$
$$= -\frac{1}{3x^{\frac{4}{3}}}$$
$$= -\frac{1}{3\sqrt[3]{x^4}}$$

Question 22

$$y = 4x^{-1} - x^2$$
$$\frac{dy}{dx} = -\frac{4}{x^2} - 2x$$
at $x = 2$
$$\frac{dy}{dx} = -\frac{4}{(2)^2} - 2(2)$$
$$= -5$$

Question 23

$$y = x^{-2}$$
$$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$$
at $x = -2$
$$\frac{dy}{dx} = -\frac{2}{(-2)^3}$$
$$= \frac{1}{4}$$

Question 24

$$y = 1 - x^{-1}$$
$$\frac{dy}{dx} = 0 - (-1)x^{-2} = \frac{1}{x^2}$$
at $x = 4$
$$\frac{dy}{dx} = \frac{1}{4^2}$$
$$= \frac{1}{16}$$

Question 25

$$y = 3x^3 - 2x^{-1}$$
$$\frac{dy}{dx} = 9x^2 + \frac{2}{x^2}$$
at $x = 1$
$$\frac{dy}{dx} = 9(1)^2 + \frac{2}{(1)^2}$$
$$= 11$$

Question 26

$$y = x^{\frac{4}{3}}$$
$$\frac{dy}{dx} = \frac{4}{3}\sqrt[3]{x}$$
at $x = 8$
$$\frac{dy}{dx} = \frac{4}{3}\sqrt[3]{8}$$
$$= \frac{8}{3}$$

Question 27

$$y = 6x^{\frac{1}{3}} + 2x^{-3}$$
$$\frac{dy}{dx} = \frac{2}{\sqrt[3]{x^2}} - \frac{6}{x^4}$$
at $x = 1$
$$\frac{dy}{dx} = \frac{2}{\sqrt[3]{(1)^2}} - \frac{6}{(1)^4}$$
$$= -4$$

Question 28

$$y = 2x^{-1} + x^2 + 16x^{-2}$$
$$\frac{dy}{dx} = -\frac{2}{x^2} + 2x - \frac{32}{x^3}$$
at $x = 2$
$$\frac{dy}{dx} = -\frac{2}{(2)^2} + 2(2) - \frac{32}{(2)^3}$$
$$= -\frac{1}{2}$$

Question 29

$$y = x^{-1}$$
$$\frac{dy}{dx} = -\frac{1}{x^2} = -\frac{1}{4}$$
$$x^2 = 4$$
$$x = \pm 2$$

at $x = -2, y = -\frac{1}{2} \Rightarrow (-2, -\frac{1}{2})$

at $x = 2, y = \frac{1}{2} \Rightarrow (2, \frac{1}{2})$

Question 30

$$y = x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$
$$1 = \frac{1}{2\sqrt{x}}$$
$$2\sqrt{x} = 1$$
$$\sqrt{x} = \frac{1}{2}$$
$$x = \frac{1}{4}$$

at $x = \frac{1}{4}, y = \sqrt{\frac{1}{4}} = \frac{1}{2} \Rightarrow (\frac{1}{4}, \frac{1}{2})$

Question 31

$$y = x^2 - 108x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2x - \frac{54}{\sqrt{x}}$$
$$0 = 2x - \frac{54}{\sqrt{x}}$$
$$2x = \frac{54}{\sqrt{x}}$$
$$2x\sqrt{x} = 54$$
$$x^{\frac{3}{2}} = 27$$
$$(x^{\frac{3}{2}})^{\frac{2}{3}} = (3^3)^{\frac{2}{3}}$$
$$x = 3^2$$
$$= 9$$
$$y = 9^2 - 108 \times 3$$
$$= -243$$
$$(9, -243)$$

Question 32

$$y = x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$
at $x = 4$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$
$$y = \frac{1}{4}x + c$$
$$2 = \frac{1}{4}(4) + c$$
$$c = 1$$
$\therefore$ equation of tangent $y = \frac{1}{4}x + 1$

Question 33

$y = x^{-1}$

$\frac{dy}{dx} = -\frac{1}{x^2}$

at $x = 1$

$\frac{dy}{dx} = -\frac{1}{(1)^2} = -1$

$y = -1x + c$

$1 = -1(1) + c$

$c = 2$

$\therefore$ equation of tangent $y = -x + 2$

Question 34

$y = x^{-2}$

$\frac{dy}{dx} = -\frac{2}{x^3}$

at $x = 2$

$\frac{dy}{dx} = -\frac{2}{(1)^3} = -\frac{1}{4}$

$y = -\frac{1}{4}x + c$

$\frac{1}{4} = -\frac{1}{4}(2) + c$

$c = \frac{3}{4}$

$\therefore$ equation of tangent $y = -\frac{1}{4}x + \frac{3}{4}$ or $4y = -x + 3$

Question 35

$$16y = 41x + 6$$

$$y = \frac{41}{16}x + \frac{6}{16} \Rightarrow m = \frac{41}{16}$$

$$y = 2x - x^{-1}$$

$$\frac{dy}{dx} = 2 + \frac{1}{x^2}$$

$$\frac{41}{16} = 2 + \frac{1}{x^2}$$

$$\frac{1}{x^2} = \frac{9}{16}$$

$$x^2 = \frac{16}{9}$$

$$x = \pm\frac{4}{3}$$

at $x = -\frac{4}{3}$

$$y = 2(-\frac{4}{3}) - \frac{1}{(-\frac{4}{3})}$$

$$= -\frac{8}{3} + \frac{3}{4}$$

$$= -\frac{23}{12}$$

at $x = \frac{4}{3}$

$$y = 2(\frac{4}{3}) - \frac{1}{(\frac{4}{3})}$$

$$= \frac{8}{3} - \frac{3}{4}$$

$$= \frac{23}{12}$$

$\therefore$ coordinates are $(-\frac{4}{3}, -\frac{23}{12})$ and $(\frac{4}{3}, \frac{23}{12})$

Question 36

$$\text{Gradient at } P(x,\frac{1}{x}) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{1}{x+h}-\frac{1}{x}\right)\frac{1}{h}$$

$$= \lim_{h\to 0}\left(\frac{x-(x+h)}{x(x+h)}\right)\frac{1}{h}$$

$$= \lim_{h\to 0}\left(\frac{h}{x(x+h)}\right)\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{1}{x(x+h)}$$

$$= \frac{1}{x^2}$$

$$\text{Gradient at } P(x,\sqrt{x}) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x}+\sqrt{x+h}}$$

$$= \frac{1}{2\sqrt{x}}$$

Miscellaneous exercise five

Question 1

a $5^4 = 5^n$

$n = 4$

b $2^4 = n$

$n = 16$

c $2^n = 8$

$= 2^3$

$n = 3$

d $6^{3+4} = 6^7$

$6^n = 6^7$

$n = 7$

e $2^6 \times 2^3 = 2^n$

$2^9 = 2^n$

$n = 9$

f $3^{2+n} = 3^6$

$2 + n = 6$

$n = 4$

g $10^{2+n} = 10^6$

$2 + n = 6$

$n = 4$

h $2^4 \times 2^3 = 2^n$

$2^7 = 2^n$

$n = 7$

i $4 \times 4^2 = 4^n$

$4^3 = 4^n$

$n = 3$

j $8^{5-n} = 8^2$

$5 - n = 2$

$n = 3$

k $n = 0$

l $3^{2+n+1} = 3^7$

$n + 3 = 7$

$n = 4$

m $\dfrac{5^9 \times 5^n}{5^3} = 5^8$

$5^{6+n} = 5^8$

$n + 6 = 8$

$n = 2$

n $\dfrac{5^9}{5^3 \times 5^n} = 5^2$

$5^{6-n} = 5^2$

$6 - n = 2$

$n = 4$

o $(2^3)^3 = 2^n$

$2^9 = 2^n$

$n = 9$

Question 2

a at $x = 4, y = 4^2 = 16$

at $x = 5, y = 5^2 = 25$

$\therefore$ average rate of change is 9

b $\dfrac{dy}{dx} = 2x$

at $x = 8, \dfrac{dy}{dx} = 2(8) = 16$

Question 3

a at $x=1, y=1^3=1$

at $x=3, y=3^3=27$

$\therefore$ average rate of change $=\dfrac{27-1}{3-1}=13$

b $\dfrac{dy}{dx}=-6x^2$

at $x=-2, \dfrac{dy}{dx}=-6(2)^2=-24$

Question 4

2, 4, 8....

a $T_{10}=(2)2^9$
$=1024$

b $T_{30}=(2)2^{29}$
$=1\,073\,741\,824$

c $S_{10}=\dfrac{2(2^{10}-1)}{2-1}$
$=2046$

d $S_{30}=\dfrac{2(2^{30}-1)}{2-1}$
$=2\,147\,483\,646$

Question 5

a $\dfrac{dy}{dx}=-3x^2$

b $\dfrac{dy}{dx}=10x-\dfrac{3}{\sqrt{x}}$

c $\dfrac{dy}{dx}=10x-\dfrac{2}{x^3}$

Question 6

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

at $x = 0.5$

$$\frac{dy}{dx} = -\frac{1}{(0.5)^2}$$

$$= -4$$

Question 7

a reciprocal relationship $xy = 6$

b symmetry of y values suggests a quadratic relationship which is confirmed by a constant second difference of 2. This indicates $a = 1$.

When $x = 0, y = 1 \Rightarrow c = 1$

$y = x^2 + bx + 1$

Using (1, 2)

$2 = 1^2 + b + 1$

$b = 0$

$\Rightarrow$ equation is $y = x^2 + 1$

c Constant first difference of 3 indicated a linear relationship with a gradient of 3.

$y = 3x + c$

When $x = 0, y = 5 \Rightarrow c = 5$

$\therefore$ equation is $y = 3x + 5$

d Constant first ratio of 3 indicates an exponential relationship with a base of 5. $y = a \times 5^x$.

Using (2, 25)

$25 = a \times 5^2$

$a = 1$

$\therefore$ equation is $y = 5^x$

e symmetry of y values suggests a quadratic relationship which is confirmed by a constant second difference of 2. This indicates $a=1$.

When $x=0, y=0 \Rightarrow c=0$

$$
\begin{aligned}
&y=x^2+bx\\
&\text{Using } (-4, 12)\\
&12=(-4)^2-4b\\
&4b=4\\
&b=1\\
&\Rightarrow \text{equation is } y=x^2+x=x(x+1)
\end{aligned}
$$

f Constant first ratio of 10 indicates an exponential relationship with a base of 10. $y=a\times 10^x$

$$
\begin{aligned}
&\text{Using } (2, 100)\\
&100=a\times 10^2\\
&\ \ a=1\\
&\therefore \text{ equation is } y=10^x
\end{aligned}
$$

g Constant first ratio of 2 indicates an exponential relationship with a base of 2. $y=a\times 2^x$

$$
\begin{aligned}
&\text{Using } (2, 16)\\
&16=a\times 2^2\\
&4a=16\\
&\ \ a=4\\
&\therefore \text{ equation is } y=4\times 2^x\\
&\qquad\qquad\qquad\quad =2^{x+2}
\end{aligned}
$$

h reciprocal relationship $xy=-24$

i The three zero y values indicate the three x-intercepts in a cubic relationship

$$
\begin{aligned}
&y=ax(x-3)(x+3)\\
&\text{Using } (1, -16)\\
&-16=-a\times 1\times(1-3)(1+3)\\
&-16=-8a\\
&\ \ a=2\\
&\therefore \text{ equation is } y=2x(x-3)(x+3)
\end{aligned}
$$

Question 8

Let the three angles be $10, 10+d, 10+2d$

$$\begin{aligned} 10+10+d+10+2d &= 180 \\ 30+3d &= 180 \\ 3d &= 150 \\ d &= 50 \end{aligned}$$

The other angles are 60° and 110°.

Question 9

$$\begin{aligned} T_4 = ar^3 &= 100 \\ a(5)^3 &= 100 \\ a &= \frac{100}{125} \\ a &= 0.8 \end{aligned}$$

$T_1 = 0.8, T_{n+1} = 5T_n$

Question 10

a $2 \times 10^7 \times 4 \times 10^4$

$= 8 \times 10^{11}$

b 8×10^{11}

c $(2 \times 10^7)^3$

$= 2^3 \times 10^{21}$

$= 8 \times 10^{21}$

d $(4 \times 10^4)^2$

$= 4^2 \times 10^8$

$= 16 \times 10^8$

$= 1.6 \times 10^9$

e $\frac{4\times10^4}{2\times10^7}$

$= 2\times10^{-3}$

f $\frac{2\times10^7}{4\times10^4}$

$= 0.5\times10^3$

$= 5\times10^2$

Question 11

Sequence 1

a 5, 17, 53, 161, 485

b Neither

c $S_5 = 5+17+53+161+485 = 721$

d $T_{18} = 774\,840\,977$ (By classpad)

e $S_{18} = 1162\,261\,446$

Sequence 2

a $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2$

b Geometric

c $S_5 = \frac{\frac{1}{8}(2^5-1)}{2-1} = 3.875$

d $T_{18} = 0.125\times2^{17} = 16384$

e $S_{18} = \frac{0.125(2^{17}-1)}{2-1} = 32767.875$

Sequence 3

a $-5, 5, 15, 25, 35$

b Arithmetic

c $$\begin{aligned} S_5 &= \frac{5}{2}[2(-5)+4(10)] \\ &= 75 \end{aligned}$$

d $$\begin{aligned} T_{18} &= (-5)+17(10) \\ &= 165 \end{aligned}$$

e $$\begin{aligned} S_{18} &= \frac{18}{2}[2(-5)+17(10)] \\ &= 1440 \end{aligned}$$

Question 12

a $$\frac{dy}{dx} = 6x^2 - 1$$

at $x = 1$

$$\begin{aligned} \frac{dy}{dx} &= 6(1)^2 - 1 \\ &= 5 \end{aligned}$$

$$\begin{aligned} y &= 5x + c \\ 4 &= 5(1) + c \\ c &= -1 \end{aligned}$$

$\therefore$ equation of tangent is $y = 5x - 1$

b $$\begin{aligned} 6x^2 - 1 &= 23 \\ 6x^2 &= 24 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

at $x = 2, y = 2(2)^3 - 2 + 3 = 17 \Rightarrow (2, 17)$

at $x = -2, y = 2(-2)^3 - (-2) + 3 = 17 \Rightarrow (-2, -11)$

Question 13

a From $x = 1$ to $x = 6$ the function has an average rate of change of 64

b At $x = 5$ the function has an instantaneous rate of change of 105

Question 14

C and D are immediate choices as the function is cubic.

As x gets larger, $\dfrac{dy}{dx}$ becomes large and positive

$\Rightarrow$ Graph C

Question 15

a

$$x(x+6)(x-6)=0$$

$$x=-6,0,6$$

$$\therefore 3 \text{ places}$$

b As $x \to \infty, \dfrac{dy}{dx}$ is positive

For example, at $x = 100$

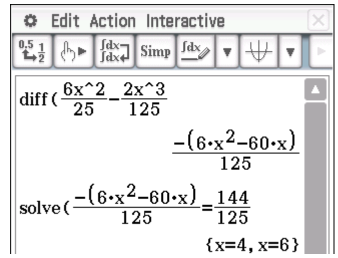
$$\frac{dy}{dx} = 100 \times 106 \times 104 > 0$$

c As $x \to -\infty, \dfrac{dy}{dx}$ is negative

For example, at $x = -100$

$$\frac{dy}{dx} = -100 \times (-94) \times (-106) < 0$$

Question 16



$$\frac{dy}{dx} = \frac{12x}{25} - \frac{6x^2}{125}$$

$$= \frac{60x - 6x^2}{125}$$

$$\frac{60x - 6x^2}{125} = \frac{144}{125}$$

$$60x - 6x^2 = 144$$

$$6x^2 - 60x + 144 = 0$$

$$6(x^2 - 10x + 24) = 0$$

$$6(x-6)(x-4) = 0$$

$$x = 4, x = 6$$

at $x = 4, y = \frac{6(4)^2}{25} - \frac{2(4)^3}{125} = 2.816$